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iMechanica: Journal of Applied Mechanics and Vocational Training
Vol. 2 | No. 1 | 2025 pp. 6–13 |**ISSN: 3066-3474**

Mathematical Modeling of Reservoir Pressure Maintenance Through Industrial Wastewater Injection

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Submitted: 1-May, 2025
Accepted: 2-June, 2025
Published: 9-July, 2025

Vol. 2, **No.** 1, 2025.
10.62762/MJAMVT.2024.000000

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Abstract

This paper presents a mathematical model for maintaining reservoir pressure through the injection of industrial wastewater in oil fields. When reservoir pressure drops significantly over time, oil production becomes inefficient and challenging. The water injection method is a widely used technique to sustain and increase reservoir pressure, thereby enhancing oil recovery. We focus on the peripheral (outside contour) water flooding method, especially suited for small to medium-sized oil and gas reservoirs. The model is formulated as a system of two-dimensional parabolic partial differential equations, accounting for the flow dynamics in both oil and water zones. The system is solved with appropriate initial and boundary conditions, transformed into a dimensionless form for computational efficiency. This study provides a theoretical basis and numerical approach to optimize pressure maintenance strategies via water injection, promoting improved oil extraction efficiency.

Keywords: Reservoir pressure maintenance, water injection, wastewater flooding, oil recovery, partial differential equations, peripheral flooding, mathematical modeling, dimensionless formulation

Introduction

In oil fields, the **“Water injection” regime** is applied to maintain or increase reservoir pressure when it becomes critically low. It is well known that over time, reservoir pressure tends to decline in oil fields. As a result, the production process slows down, and oil recovery becomes increasingly difficult. In such cases, the **water flooding method** is employed to restore reservoir energy. By injecting water into the reservoir, pressure can be increased, which over time leads to a partial recovery of oil production.

There are several methods of water injection used for oil displacement, including:

- **Peripheral (contour) flooding,**
- **Pattern flooding (injection from within the contour),**
- **bottom-up flooding.**

These techniques help improve oil recovery efficiency by maintaining adequate reservoir pressure.

Peripheral (outside contour) water flooding method.This method is primarily used for relatively small-scale oil or gas reservoirs. It involves injecting water through injection wells (known as “nagnetatel wells”) located close to the outer contour (boundary) of the oil field — typically at a distance of 50 to 100 meters. The production wells are positioned along an inner contour, parallel to the outer boundary. As water is injected into the reservoir, over time, the pressure in the oil-producing zone increases. This pressure buildup makes it possible to extract oil more effectively and in sufficient quantities.

A schematic diagram of the peripheral water flooding method is shown in Figure 2.1.

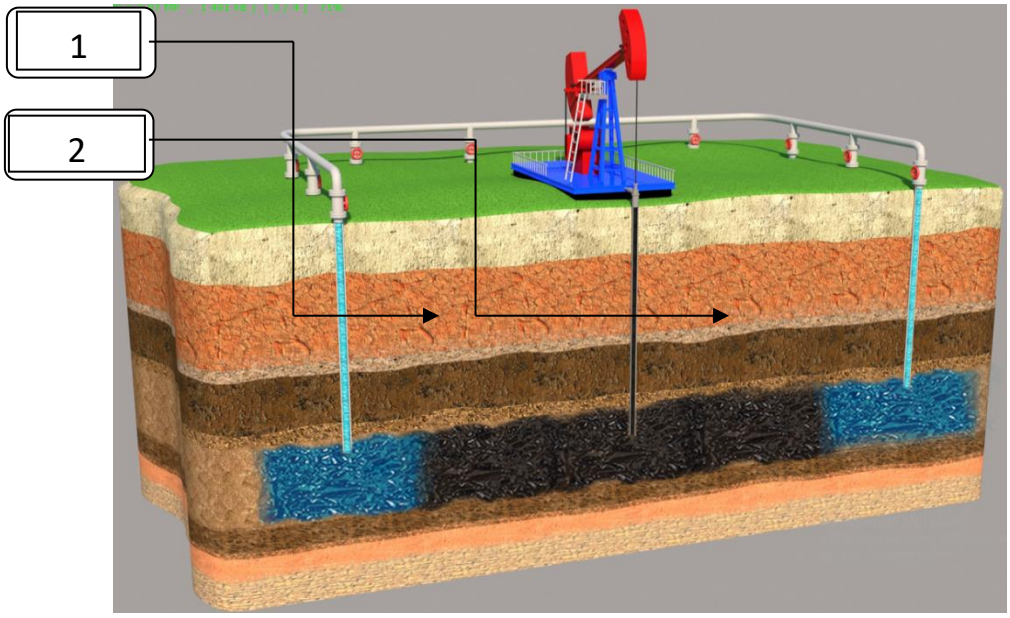


Fig.1. Peripheral (outside-contour) water flooding method for oil recovery.
Wells: 1 – Water injection well, 2 – Oil production well

This assumption-based mathematical model is expressed in the form of a system of partial differential equations of parabolic type in two spatial dimensions and is written as follows:

$$\begin{cases} \frac{\partial}{\partial x}\left(\frac{k}{\mu_1}\frac{\partial P_1}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{k}{\mu_1}\frac{\partial P_1}{\partial y}\right)=a\beta_H\frac{\partial P_1}{\partial t}-Q\ beside\ (x,y()_1) \\ \frac{\partial}{\partial x}\left(\frac{k}{\mu_2}\frac{\partial P_2}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{k}{\mu_2}\frac{\partial P_2}{\partial y}\right)=(1-a)\beta_B\frac{\partial P_2}{\partial t}+Q\ beside\ (x,y()_2) \end{cases} \quad (1)$$

This system of differential equations (2.23) is considered with the following initial and boundary conditions:

$$P_1=P_2=P_H(x,y)\ beside=0,(x,y)\in G_1+G_2, \quad (2)$$

$$q_{\mathrm{H}i_q}(t)=\oint_{s_{i_q}}-\frac{k}{\mu_1}\frac{\partial P_1}{\partial n}beside\ (x,y)\in s_{i_q},i_q=\overline{1,N_q}, \quad (3)$$

$$q_{\mathrm{B}i_q}(t)=\oint_{s_{i_q}}\frac{k}{\mu_2}\frac{\partial P_1}{\partial n}beside\ (x,y)\in s_{i_q},i_q=\overline{1,M_q}, \quad (4)$$

$$\frac{\partial P_2}{\partial n} = 0 \; beside \; (x,y) \in \Gamma_2, \tag{5}$$

$$P_1(x,y) = P_2(x,y), \frac{k}{\mu_1}\frac{\partial P_1}{\partial n} = \frac{k}{\mu_2}\frac{\partial P_2}{\partial n} \; beside \; (x,y) \in \Gamma_1, \tag{6}$$

$$\frac{\partial l(x,y,t)}{\partial t} = -\frac{k}{\mu_1 m(a-a_0)}\frac{\partial P_1(x,y,t)}{\partial n}, \tag{7}$$

$$l(x,y,0) = \phi(x,y) beside \; (x,y) \in \Gamma_1. \tag{8}$$

Here,

n – inward normal to the boundary,

q – flow rate (production rate) of the i-th oil well,

Nq – number of wells,

S – contour of the i-th well,

c_1 – dimensional scaling coefficient,

p_1, – pressures in the oil and water zones of the reservoir,

k – permeability coefficient of the reservoir,

m– porosity coefficient,

h – reservoir thickness,

n_1– normal to the contour s.

To solve the given problem, we transform the mathematical model and the boundary conditions into a dimensionless form. For this purpose, we introduce the following notations:

$$x^* = \frac{x}{L}; \quad y^* = \frac{y}{L}; \quad k_p{}^* = \frac{k_p}{k_x}; \quad P_1^* = \frac{P_1}{P_x}; \quad P_2^* = \frac{P_2}{P_x};$$

$$t = \frac{\beta_H a \mu_1 L^2}{k_x}\tau; \quad q_H^* = \frac{q_H \mu_1}{\pi k_x P_x h_x}; \quad q_B^* = \frac{q_B \mu_2}{\pi k_x P_x h_x}; \quad R^* = \frac{(a-a_0)\beta_B}{a\beta_H}.$$

Here,

P_x– characteristic value of the reservoir pressure,

k_x– characteristic value of the reservoir permeability,

L– characteristic length of the oil reservoir.

Taking into account the above notations and omitting the asterisk symbol (*), the dimensionless form of the mathematical model with the given boundary conditions can be written as follows:

$$\frac{\partial}{\partial x}\left(\frac{k_1 h_1}{\mu_1}\frac{\partial p_1}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{k_1 h_1}{\mu_1}\frac{\partial p_1}{\partial y}\right) = 2\alpha\beta h_1 \frac{\partial p_1}{\partial t}, \; (x,y) \in G_1$$

$$\frac{\partial}{\partial x}\left(\frac{k_2 h_2}{\mu_2}\frac{\partial p_2}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{k_2 h_2}{\mu_2}\frac{\partial p_2}{\partial y}\right) = (1-\alpha_{ocm})\beta h_2 \frac{\partial p_2}{\partial t}, \; (x,y) \in G_2 \tag{9}$$

$$t=0; \quad p_1 = p_2 = p_H(x,y), \quad (x,y) \in G_1 U G_2 \tag{10}$$

$$c_1 q_N^{(t)} = \oint_{S_i} \frac{k_1 h_1}{\mu_1} \frac{\partial p_1}{\partial n_1} ds, \quad (x,y) \in S_i, \quad i = \overline{1, N_Q} \tag{11}$$

$$q_{Bi_q}(t) = \int_{S_{i_q}} \frac{k}{\mu_2} \frac{\partial P_1}{\partial n} ds \ \ npu \ (x,y) \in s_{i_q}, \quad i_q = \overline{1, M_q}, \tag{12}$$

$$\frac{\partial p_2}{\partial n_3} = 0, \qquad (x,y) \in \Gamma_2 \tag{13}$$

$$p_1(x,y) = p_2(x,y), \quad (x,y) \in \Gamma_1 \tag{14}$$

$$\frac{k_1}{\mu_1} \frac{\partial p_1}{\partial n_2} = \frac{k_2}{\mu_2} \frac{\partial p_2}{\partial n_2}, \quad (x,y) \in \Gamma_1 \tag{15}$$

$$-\frac{\partial l}{\partial t} = \frac{k_2}{\mu_2 m(\alpha - \alpha_{ocm})} \frac{\partial p_2}{\partial n_2}; \quad l(0) = \delta(x,y), \ (x,y) \in \Gamma_1 \tag{16}$$

To discretize the two-dimensional boundary value problem (9) – (16), which describes the filtration process in a porous medium of an oil reservoir, we apply the operator splitting scheme (also known as the method of variable directions). This is combined with the differential sweep method (progonka), as described in [2].

This approach allows the use of the differential sweep method along each coordinate direction. Specifically, within each time interval: First, at time *t+0.5t* we apply the sweep method along the x-direction for each fixed y, obtaining an intermediate solution. Then, at time *t+1t* we apply the sweep method along the y-direction for each fixed x, using the intermediate solution to compute the final value for the time step.

This algorithm is then repeated at each subsequent time step. To implement this, a uniform computational grid is constructed over the filtration domain

$$\Omega_\delta = \{(x_i = ih, \ y_j = jh); \ i = 1,2,\ldots, N_j; \ j = 1,2,\ldots, M \ \},$$

$$\begin{cases} \dfrac{\partial}{\partial x}\left(k_{1ij} \dfrac{\partial p_{1j}^{(k+0.5)}(x)}{\partial x}\right) - \dfrac{1}{0.5\tau} p_{1j}^{(k+0.5)}(x) = -\dfrac{p_{1j}^{(k)}(x)}{0.5\tau} - \Lambda\left[k_{1ij} p_{1ij}^{(k)}\right] \\ \dfrac{\partial}{\partial y}\left(k_{1ij} \dfrac{\partial p_{1i}^{(k+1)}(y)}{\partial y}\right) - \dfrac{1}{0.5\tau} p_{1i}^{(k+1)}(y) = -\dfrac{p_{1i}^{(k+0.5)}(y)}{0.5\tau} - \Lambda\left[k_{1ij} p_{1ij}^{(k+0.5)}\right] \\ \dfrac{\partial}{\partial x}\left(k_{2ij} \dfrac{\partial p_{2j}^{(k+0.5)}(x)}{\partial x}\right) - \dfrac{R^*}{0.5\tau} p_{2j}^{(k+0.5)}(x) = -\dfrac{p_{2j}^{(k)}(x)}{0.5\tau} - \Lambda\left[k_{2ij} p_{2ij}^{(k)}\right] \\ \dfrac{\partial}{\partial y}\left(k_{2ij} \dfrac{\partial p_{2i}^{(k+1)}(y)}{\partial y}\right) - \dfrac{R^*}{0.5\tau} p_{2i}^{(k+1)}(y) = -\dfrac{p_{2i}^{(k+0.5)}(y)}{0.5\tau} - \Lambda\left[k_{2ij} p_{2ij}^{(k+0.5)}\right]. \end{cases} \tag{17}$$

where:

$$k_1 = \frac{k_p}{\mu_1};\ \ k_2 = \frac{k_p}{\mu_2};$$

$$\Lambda\left[k_{1ij}p_{1ij}^{(k)}\right] = \frac{k_{1i-0.5,j}p_{i-1,j}^{(k)} - (k_{1i-0.5,j} + k_{1i+0.5,j})p_{ij}^{(k)} + k_{1i+0.5,j}p_{i+1,j}^{(k)}}{h_{xy}^2},$$

$$\Lambda\left[k_{1ij}p_{1ij}^{(k+0.5)}\right] == \frac{k_{1i,j-0.5}p_{ij-1}^{(k+0.5)} - (k_{1i,j-0.5} + k_{1ij+0.5j})p_{ij}^{(k+0.5)} + k_{i,j+0.5}p_{i,j+1}^{(k+0.5)}}{h_{xy}^2},$$

Pressure at the k-th time step

Pressure at the intermediate time step *k+0.5k*

The finite difference system of differential equations (17) is solved at each *xi* direction using the differential sweep method (progonka) for the time layer *k+0.5k,* subject to the given initial conditions. Similarly, for the next time layer k+1k, the solution is computed along each yj direction, using the results obtained at time layer *k+0.5k* as the initial condition. Based on the idea of the differential sweep method, the general solution of the first equation of the finite difference problem (17), considering the boundary conditions, can be written as follows:

$$P_i(x) = \frac{\gamma_i(x)U_i(x) - \alpha_i(x)w_i(x)}{\alpha_i(x)v_i(x) - \beta_i(x)U_i(x)};$$

$$\frac{dP_i}{dx} = \frac{1}{\sigma}\cdot\frac{\gamma_i(x)v_i(x) - \beta_i(x)w_i(x)}{\alpha_i(x)v_i(x) - \beta_i(x)U_i(x)}. \qquad (18)$$

This solution provides the values of the pressure function and its first-order derivatives at all points along the unknown *x*-direction, for every point in the y-direction. Here, the right and left sweep coefficients — denoted as *u(x) v(x),* and *w(x),* and their corresponding values — are determined as the solution of a Cauchy problem. For the left sweep coefficients, the Cauchy problem is formulated as follows:

$$\left\{k_i\frac{du_i}{dx} = v_i, \frac{dv_i}{dx} = Q_iu_i, \frac{dw_i}{dx} = R_iu_i\right. . \qquad (19)$$

The initial conditions for this system of equations are determined from the boundary condition (2.33) on the left side of the filtration domain.

$$u_1 = u_{10}/k_1,\ \ v_1 = v_{10}\ ,\ \ w_1 = w_{10} \qquad (20)$$

The Cauchy problem for the right sweep (progonka) coefficients is formulated as follows:

$$\begin{cases} k_i \dfrac{d\alpha_i}{dx} = \beta_i \\ \dfrac{d\beta_i}{dx} = Q_i \alpha_i \\ \dfrac{d\gamma_i}{dx} = R_i \alpha_i \end{cases} \tag{21}$$

The initial conditions for this system of equations are determined from the boundary condition (21) on the **right side** of the filtration domain.

$$\alpha_n = \alpha_{10} / k_n, \quad \beta_{n1} = \beta_{n0}, \ \gamma_{n1} = \gamma_{n0} \tag{22}$$

Similarly, based on the idea of the differential sweep (progonka) method, the general solution of the second equation of the finite difference problem (2.39), taking into account the boundary conditions, can be written in the same way as above.

$$P_j(y) = \frac{\gamma_j(y) U_j(y) - \alpha_j(y) w_j(y)}{\alpha_j(y) v_j(y) - \beta_j(y) u_j(y)};$$

$$\frac{dP_j}{dy} = \frac{1}{\sigma} \cdot \frac{\gamma_j(y) v_j(y) - \beta_j(y) w_j(y)}{\alpha_j(y) v_j(y) - \beta_j(y) u_j(y)}. \tag{23}$$

The solution (23) provides the values of the pressure function and its first-order derivatives at all points along the unknown *y*-direction for every point in the *x*-direction. Here, the right and left differential sweep coefficients, denoted by *u(y) v(y),* and *w(y),* and their corresponding values, are determined as the solution of a Cauchy problem. The Cauchy problem for the left sweep coefficients is formulated as follows:

$$\begin{cases} k_i \dfrac{du_i}{dy} = v_i \\ \dfrac{dv_i}{dy} = Q_i u_i \\ \dfrac{dw_i}{dy} = R_i u_i \end{cases} \tag{24}$$

The initial conditions for this system of differential equations are determined from the boundary condition on the left side of the filtration domain.

$$u_1 = u_{10} / k_1, \ \ v_1 = v_{10}, \ \ w_1 = w_{10} \tag{25}$$

The Cauchy problem for the **right sweep coefficients** is formulated as follows:

$$\begin{cases} k_i \dfrac{d\alpha_i}{dy} = \beta_i \\ \dfrac{d\beta_i}{dy} = Q_i \alpha_i \\ \dfrac{d\gamma_i}{dy} = R_i \alpha_i \end{cases} \quad (26)$$

The initial conditions for this system of differential equations are determined from the boundary condition on the **right side** of the filtration domain.

$$\alpha_n = \alpha_{10} / k_n, \quad \beta_{n1} = \beta_{n0}, \ \gamma_{n1} = \gamma_{n0} \quad (27)$$

Implementing this obtained discrete model on a computer does not pose significant difficulties. It can be carried out in the following two-step sequence:

Step 1: The initial conditions for the Cauchy problem are determined from the boundary conditions of the problem. The Cauchy problem with initial conditions is solved along all directions of the *x*-axis using the Runge-Kutta or the Kutta-Merson procedure, moving from left to right. Similarly, the Cauchy problem with initial conditions is solved along all directions of the xxx-axis using the Runge-Kutta or Kutta-Merson procedure, moving from right to left.

As a result, the values of the pressure function *Pi(x)* and its first-order derivatives are determined for all directions of the unknown *x* at every point in the *y*-variable. Thus, the values of the pressure function at the half time step are obtained for all points in the oil reservoir domain

Step 2: We advance by a half time step and compute the pressure function values at time step k+1k for all points in the oil reservoir domain. This step is performed similarly to Step 1. The process of solving the Cauchy problem using the Runge - Kutta method is carried out together with applying the sweep coefficients and, at each step, the normalization procedure for these coefficients.

At each step, when calculating the given vector U_i in the Cauchy problem for the left side, the normalized vector $U = (u, v, w)$ is used instead of the vector U_i from the right side. Similarly, when solving the Cauchy problem from left to right, the normalized vector $A = (\alpha, \beta, \gamma)$ is used instead of the vector Ai from the right side while computing the vector $Ai–1A$.The normalization procedure is not applied if the chosen method preserves stability. To determine the position of the phase-separating boundary, formula (*22.24.27*) is used. The application of the differential sweep method automatically accounts for the continuity conditions of the phase flows and the pressure's non-discontinuity during the solution process.

Thus, when solving a two-dimensional differential equation boundary value problem

with a moving boundary, reducing the problem to a finite difference formulation using a cross-section scheme and solving it with the differential sweep method does not pose significant difficulties.

Conclusion

The developed mathematical model effectively captures the dynamics of reservoir pressure maintenance using the peripheral water injection method. By incorporating the system of parabolic partial differential equations and suitable initial and boundary conditions, the model simulates the interaction between oil and water phases within the reservoir. Transforming the problem into dimensionless variables simplifies numerical implementation and enhances computational stability.

This approach can serve as a robust framework for optimizing injection strategies in oil fields, leading to improved oil recovery and prolonged reservoir life. Future work may extend the model to more complex reservoir conditions and different injection techniques.

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